Lecture 8: Recap:

Definition: (Discrete Fourier Transform) Given $f_0, f_1, \ldots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)}$$

for $k = 0, 1, 2, \ldots, n-1$

The inverse discrete Fourier Transform recovers the original signal:

$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \ldots, n-1$$

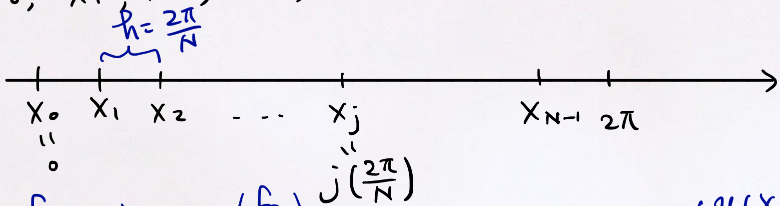
Recall:

Consider: $\dfrac{d^2 u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition.

$u(0) = u(2\pi)$

Suppose f is measured only at N discrete points:

$x_0, x_1, x_2, \ldots, x_{N-1}$



(unknown)

Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \overset{def}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

Then:

$$\begin{pmatrix} u''(x_0)\\ u''(x_1)\\ \vdots \\ u''(x_{N-1})\end{pmatrix} \approx \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & 0 & 1\\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & 0 & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

$M_{N\times N}(\mathbb{R})$

(using the fact $u_0 = u_N$, $u_{-1} = u_{N-1}$)

Conclusion:

$$\boxed{\frac{d^2u}{dx^2} = f \quad \text{can be discretized as} \quad \boxed{\tilde{D}\vec{u} = \vec{f}} \quad \text{(Linear System)}}$$

Thus, $\frac{d^2u}{dx^2} = f$ can be discretized as $\boxed{\tilde{D}\vec{u} = \vec{f}}$ (Linear System)

<u>Remark:</u> Note that $\tilde{D}$ can be a very BIG matrix.

<u>Goal:</u> Design the numerical spectral method. We need to:

① Determine eigenvalues / eigenvectors of $\tilde{D}$

② Rank of $\tilde{D}$ (to understand the sol. of linear system)

In continuous case, e^{ikx} is an eigenfunction of $\frac{d^2}{dx^2}$, that is periodic.

In discrete case, define:

$$\underset{\substack{\cap \\ \mathbb{C}^N}}{\overrightarrow{e^{ikx}}} \underline{\underline{\text{def}}} \begin{pmatrix} e^{ikx_0} \\ e^{ikx_1} \\ \vdots \\ e^{ikx_{N-1}} \end{pmatrix}$$

(Capture the values of e^{ikx} at N discrete points)

Claim: $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$

Proof: For each j,

$$\left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_j = \frac{e^{ikx_{j-1}} - 2e^{ikx_j} + e^{ikx_{j+1}}}{h^2}, \qquad x_{j-1} = x_j - h, \quad x_{j+1} = x_j + h$$

$$= e^{ikx_j}\left(\frac{e^{-ikh} - 2 + e^{ikh}}{h^2}\right)$$

$$= e^{ikx_j}\left(\frac{\cos kh - i\cancel{\sin kh} - 2 + \cos kh + i\cancel{\sin kh}}{h^2}\right)$$

$$= e^{ikx_j}\left(\frac{\overset{1-2\sin^2\frac{kh}{2}}{2\cos kh} - 2}{h^2}\right) = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{-\lambda_k^2} e^{ikx_j}$$

Let $-\lambda_k^2 = \left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)$. Then: $\tilde{D}\,\overrightarrow{e^{ikx}} = -\lambda_k^2\,\overrightarrow{e^{ikx}}$.

Claim: $\{\overrightarrow{e^{ikx}}\}_{k=0}^{N-1}$ is a basis of $\mathbb{C}^N$ (consisting of eigenvectors)

Pf:

$$\begin{pmatrix} \Big| & \Big| & & \Big| \\ \overrightarrow{e^{i0x}} & \overrightarrow{e^{i(1)x}} & \cdots & \overrightarrow{e^{i(N-1)x}} \\ \Big| & \Big| & & \Big| \end{pmatrix} = \underset{\sim}{A_\omega} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix} ; \ \omega = e^{i\frac{2\pi}{N}}$$

Numerical Spectral method

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{N-1}$ is a basis. We can write:

$$\vec{u} = \sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \quad \text{and} \quad \vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$\hat{u}_k \in \mathbb{C}$

In other words, for each j, $f_j = f(x_j) = \sum_{k=0}^{N-1} \hat{f}_k \left(\overrightarrow{e^{ikx}}\right)_j$

$$= \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_j}, \quad x_j = \frac{2\pi j}{n}$$

(DFT!)

$\therefore$ $\hat{f}_k$ can be determined by DFT.

To solve $\frac{d^2u}{dx^2} = f$, we approximate it by

$$\tilde{D}\vec{u} = \vec{f}.$$

Now, $\tilde{D}\vec{u} = \vec{f}$ becomes:

$$\tilde{D}\left(\sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}}\right) = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k \underbrace{\tilde{D}\overrightarrow{e^{ikx}}}_{\parallel \; (-\lambda_k^2)\overrightarrow{e^{ikx}}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k (-\lambda_k^2) \overrightarrow{e^{ikx}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

Comparing coefficients, we get

$$\underbrace{-\lambda_k^2}_{\text{known}} \underbrace{\hat{u}_k}_{\text{unknown}} = \underbrace{\hat{f}_k}_{\text{known}} \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

(algebraic equation)